ii) he image of T' is contained in the hand [u'] generated by a suitable $a' \in E'$. The two possibilities listed above are disjoint if E' is in addition weakly sequentially

FACTORING WEAKLY COMPACT OPERATORS

Proof. Suppose that ii) fails. Then there are a uncountable family of pairwise disform elements $\hat{u}_x' \in E'$ a **UDZEJUDJ** u_y **MITMATZNOD** v_y that

The aim of this note is to discuss the structure of weakly compact operators. We extend results well known for Banach spaces to operators acting on Banach spaces.

Let E and F be two Banach spaces and let $T \in L(E, F)$. Consider the follo-

wing two conditions on T:

(wP) T is weakly precompact, i.e. T maps bounded sequences into sequences with weak Cauchy subsequences;

(D) T maps weak Cauchy sequences into weakly convergent sequences.

Each weakly compact operator verifies (wP) and (D) and the product of an operator having property (D) with an weakly precompact operator is weakly compact. Moreover, the main result in [1] asserts that indeed each weakly compact operator can be obtained in such a way, so the interest in studying the two above classes (actually ideals in the sense of Pietsch) of operators.

H. P. Rosenthal [11] has obtained a nice characterization of weakly precom-

1. Theorem. An operator $T \in L(E, F)$ is weakly precompact if and only if T does not fix a copy of t_1 . A thin will be described as a substantial definition of the description of

T is said to fix a copy of the Banach space X provided that T is an isomorphism

when restricted to some subspace of E, isomorphic to X2 n xil ton 200h T (ii

the composition Soll is compacts to a (A 'T) Quatails.

Other characterizations can be obtained by easy modifications of some results due to Pelczynski [8] who considered only the case when T is the identity of a separable Banach space.

2. Theorem. Let E be a separable Banach space, F a Banach space and $T \in L(E, F)$. Then the following assertions are equivalent:

i) T fixes a copy of li; provided that I to you a mislimoolog cook I (i.e.

ii) T' fixes a copy of C[0, 1]: to know between lamon a minimon lon soob Z (ii

iii) T' fixes a copy of $l_1(\Gamma)$ for some uncountable set Γ n inlines len cook A (iii)

The proof of i) \Rightarrow ii) follows from [8], while iii) \Rightarrow i) can be adapted from [13]. H. P. Rosenthal [10] has obtained an interesting dichotomy for subspaces A of a space $L_1(\mu)$, related to condition iii) in Theorem 2 above: either there exists an $f \in L_1(\mu)$ such that $A \subset L_1(\lambda)$, where $d\lambda = f d\mu$, or A contains a subspace complemented in $L_1(\mu)$ and isomorphic to $l_1(\Gamma)$ for some uncountable set Γ . This result has also an operatorial companion.

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3. Theorem. Let E be a Banach lattice with a weak order unit u > 0, F a Banach space and $T \in L(E, F)$. Then $(ABE) \times ADE = AD$

i) either T' fix a copy of $l_1(\Gamma)$ for some uncountable set Γ ; or

ii) the image of T' is contained in the band [u'] generated by a suitable $u' \in E'$. The two possibilities listed above are disjoint if E' is in addition weakly sequentially complete.

See also [6] for a more precise result.

Proof. Suppose that ii) fails. Then there are a uncountable family of pairwise disjoint elements $u'_{\gamma} \in E'$ and a family $y'_{\gamma} \in F'$, $\gamma \in \Gamma$, such that

$$P_{\gamma}(T'y'_{\gamma}) \neq 0.$$

Here P_{γ} denotes the canonical projection of E' onto $[u'_{\gamma}]$. Clearly, we may assume that $||y_{Y}|| = ||u_{Y}|| = 1$ and (by passing to a uncountable subset of Γ if necesspaces, a spaces, a spaces, a spaces, a spaces, a spaces, a spaces, and F be two Banach $\delta < |u(v', v', u', t')|^2 + C$ sary) that

$$P_{\gamma}(T'y'_{\gamma})u > \delta$$

for each γ . Consider the operator $S: E' \to l_1(\Gamma)$ given by

$$Sx' = \{P_{\mathbf{x}}(x')u\}_{\mathbf{y}}, x' \in E'.$$

Then by Lemma 1.1 in [10], $S \cdot T'$ is an isomorphism when restricted to some sub-

space of F' isomorphic to $l_1(\Gamma)$. (4x) solution to be a morphic to $l_1(\Gamma)$ weakly sequentially complete then the spaces [u'] are weakly compactly generated, while $l_1(\Gamma)$ is not if Γ is uncountable. Jovods, owi out

If E is a separable Banach lattice with E' weakly sequentially complete then the subspaces of E' contained in subspaces [u'] are precisely the separable ones. This fact combined with the main result of [5] and Theorems 2 and 3 above provides new characterizations of weakly precompact operators:

4. Theorem. Let E be a separable Banach lattice with E' weakly sequentially complete, F a Banach space and $T \in L(E, F)$. Then the following assertions are equivalent misli) This weakly precompact; out X sough aband of the year of bins at

when restricted to some subspace of E, i; [1,0]2 to your some subspace of E, i; [1,0]2 to your

Other characterizations can be obtained by easy mo. aldarages si 'T mI (iii ts

 $\in L(E, F)$. Consider the follo-

As a corollary we reobtain Lotz's characterization of dual Banach lattices having the Radon-Nikodym property: parable Banach space.

5. Corollary. The following assertions are equivalent for E a separable Banach L(E, F), Then the following assertions are equivalent: lattice:

i) T fixes a copy of L:

i) E does not contain a copy of l1;

ii) E does not contain a complemented copy of C[0, 1]; to you a sould T (iii) E does not contain a copy of C[0, 1]; most rof (1), to you a sould T (iii)

iv) E does not contain a copy of $L_1[0, 1]$; [8] v) E is separable; E is E is weakly compactly generated. It bounded as E [01] failured E . It

of a space $T_1(t)$ related to condition iii) in Theorem 2 above: either there exists in

The restrictions on E in Theorem 4 above cannot be dropped without additional hypotheses on T. Here are two counterexamples, a nonzerbas (y) A in between (y)

The first one concerns the separability of E. Consider for T the identity of $\alpha(\Gamma)$ where Γ is a uncountable set. Then E is nonseparable (but has a strong order unit), E' is weakly sequentially complete, T is weakly precompact and Im T' is 1980 Mathematics Subject Classification. Primary 46 B. 10, 46 B. 39, 47 B 55 . . . sldarageanon est tilme Piconan, Symmery 46 A 10

The second counterexample concerns the restrictions on E'. R. C. James has constructed in [3] an example of a separable Banach space JT with nonseparable dual and such that each infinite dimensional subspace of JT contains a copy of la. Let $\pi: l_1 \to JT$ an onto mapping. Then π is weakly precompact, defined on a

separable Banach lattice and Im π' is nonseparable. does not $xT \leftarrow (un \land x)T$ bas

There is yet another characterization of weakly precompact operators due to Odell, important in producing compact operators. An operator from a Banach space to another is called a Dunford-Pettis (D. P.) operator provided it maps weak Cauchy sequences into norm convergent sequences. Odell's characterization asserts that an operator $T \in L(E, F)$ is weakly precompact iff for each D.P. operator $S \in L(F, G)$, the composition So T is compact. See [12] for details. O mercent lo sluser and

The condition (D) was first considered by Grothendieck [2] who obtained con-

ditions under which an operator verifying (D) is weakly compact. If his him

If an operator T has property (D) then T does not fix a copy of equand verifies also the Pelczynski's property (u), i.e. for each weak Cauchy sequence $(x_n)_n \subset E$ there is a weakly summable sequence $(y_n)_n \subset \overline{\text{Im } T}$ such that

B. Gerollary. Let E be a
$$\sigma$$
-complete Bannsh lattice with a weak order unit a F a Banach space and $T \in L(E, 0, \frac{W}{K}) = \frac{1}{N} \frac{1}{N} \frac{1}{N} \frac{1}{N}$

The later two conditions are independent. Consider for example the identity of e_0 and the identity of the James'space J (which fails (u)). However, under additional hypotheses the fact that T does not fix a copy of e_0 implies that T has The next result extends a well known fact due to Pelczynskin C. (C) viragorq

6. Theorem. Let $T \in L(E, F)$ an operator which does not fix a copy of e_0 . Then T

has the property (D) in each of the following cases: d A han a ball and another or

i) E is isomorphic to an AM-space;

collowing assertions are equivalent ii) F does not contain a copy of e_0 and T has property (u);

iii) E is isomorphic to a Banach lattice with an order continuous norm.

The following question is open: Let E be a Banach lattice and $T \in L(E, E)$ an operator wich does not fix a copy of eo or to Is To weakly compact ? side romes no

Here is an example which shows that T need not be weakly compact. Let

Proof. Clearly, we have only to show that
$$\Gamma$$
 includes the weakly compact. Let $(y_n)_n$ be a basic section of I includes I included and I included an inclusion of I included and I included an inclusion of I included and I included an included an included and I included an included an included an included and I included an includ

The property (D) was studied in [7] in connection with the following two classes of operators T defined on Banach lattices E and taking values in arbitrary Banach

spaces F:

T is said to be of type A provided that T is order o-continuous, i.e. of T A same to si0 $\leq x_n \downarrow$ in E implies $(Tx_n)_n$ is norm convergent in F_n

T is said to be of type B provided that $0 \le x_n \uparrow$, $||x_n|| \le K$ in E implies $(Tx_n)_n$ is norm convergent in F.

Property (D) implies type B which in turn implies type A. It was noted that property (D) (called there strong type B) is equivalent to the fact that T'' maps the band B (generated by E in E'') into F.

In the sequel we shall study the duality between (wP) and type A. and doing a lo H T' is of type A then by Corollary 7 above T" | E maps bounded sequences into

7. Theorem. Let E be a σ -complete Banach lattice with a weak order unit u > 0, F a Banach space and $T \in L(E, F)$.

Then T is of type A iff T can be factored through a weakly compactly generated Banach space. (in fact, \$\langle D_6 \) which implies (see Lemma 3.1 in [7]) that T is of ty

Proof. Suppose that T is of type A. Then, as noted in [7], T maps each order interval into a relatively weakly compact subset of F a and thus $X = \operatorname{Span} T[-u, u]$ is a weakly compactly generated subspace of F. On the other hand $x = \sup(x \land nu)$ and $T(x \land nu) \rightarrow Tx$ for each $x \in E^+$. Consequently $X \supset T(E)$. decreed subsequently

Conversely, if T can be factored through a weakly compactly generated space then T does not fix a copy of ℓ_{∞} . Indeed, ℓ_{∞} is not weakly compactly generated and any complemented subspace of a weakly compactly generated Banach space so is weakly compactly generated. Consequently (see [7], Lemma 3.1) T is of type A.

The result of Theorem 6 fails if we drop the assumption on the existence of a weak order unit. See the case when $T=1_{i}(\Gamma)$ for Γ a uncountable set.

Amir and Lindenstrauss have proved (see [1] for details) that the unit ball of the dual of a weakly compactly generated Banach space is w'-sequentially compact. also the Pelezyuski's property (a), i.e. for each weak Cauchy sequencylfraupsenco there is a weakly summable sequence (m), C Im T such that

8. Corollary. Let E be a σ -complete Banach lattice with a weak order unit u > 0, F a Banach space and $T \in L(E, F)$.

Then T is of type A iff T' maps bounded sequences into sequences with w'-convergent of a and the identity of the James space I (which fails (u)). However, asonsupsedue tional hypotheses the fact that T does not fix a copy of a implies that T has

The next result extends a well known fact due to Pelczynski. S. Theorem. Let T = L(E, F) an operator which aves not fix a copy of &

9. Proposition. Let E and F be two Banach spaces and $T \in L(E, F)$. Then the following assertions are equivalent

i) T' fixes a copy of $e_0^{(1)}$; and I have so to equal to include the solution such I (ii)

iii) E is isomorphic to a Banack tallice with an order

- ii) T' fixes a copy of l_{∞} ;
- iii) There is a complemented subspace X of E, isomorphic to l₁, such that T | X is an isomorphism and T(X) is complemented in F: 100 a sold for a target as Here is an example which shows that T need not be weakly compact. Let
- *Proof.* Clearly, we have only to show that i) \Rightarrow iii). For, let $(y'_n)_n$ be a basic sequence in F' which is equivalent to the natural basis of co and let i le the canonical inclusion of $Y = \overline{\operatorname{Span}(y'_n)_n}$ into F'. If F' of is an isomorphism then $i' \circ T'' \mid E$ verifies the assumptions of Lemma 1.1 in [10], which yields the required X.

de We comes now to the duality between (wP) and type A. benileb T. grolaredo.le

- 10. Theorem. Let E and E be two Banach lattices and T ∈ L(E, F). of bigg of T
- i) If F' has a weak order unit then T is weakly precompact iff T' is of type A.
- ii) If E is σ-complete and F' has a weak order unit then T is weakly precompact Property (D) implies type B which in turn implies Lyl squffo are of type A. vi said in turn implies type B. which in turn implies type B.

Proof. i) If T is weakly precompact then by Proposition 8 above T' fix no copy of l_{∞} , which implies (see Lemma 3.1 in [7]) that T' is of type A. we have an in [7]

If T' is of type A then by Corollary 7 above $T'' \mid E$ maps bounded sequences into sequences with w'-convergent subsequences in F'', i.e. into weak Cauchy sequences Es Banach space and T & L(E, E).

ii) If E is σ -complete and T is weakly precompact then T fix no copy of ℓ_∞ (in fact, $l_{\infty} \supset l_{1}$) which implies (see Lemma 3.1 in [7]) that T is of type A.

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3) $||x+y||p| < ||x||p| + ||y||p|, x, y \in X$, and moreover, X is topologically complete.

A basis $(x)_{i=1}^{\infty}$ in X is called a symmetric basis if $\|\sum_{i=1}^{\infty} x_i x_i\| = \|\sum_{i=1}^{\infty} \varepsilon_i x_i x_i\|$

all permutations a of N and all choices e, = -A p-Banach space X which is in the same time a vector lattice is called a p con-

vex p-Banach laffice if $|x| \le |y|$ implies $|x| \le |y|$ for all $x, y \in X$ and $|\sum |x| p$

superagic Bannels and to Fig. From Banach for all $x_0, \ldots, x_n \in X$. (Here $\sum |x_n|^n |x_n|^n$

We need the following auxiliary proposition.

Proposition 1. Let X be a p-Banach space with a symmetric basis $(x_n)_n^{\infty}$ is a p-convex p-Banach lattice (with respect to the order relation and used by the basis) and let Y be an arbitrary p Banach space. If P is a continuous projection onto Y. then $(Y \oplus Y \oplus \dots)_X \approx (P Y \oplus PY \oplus \dots)_X \oplus ((I + P)Y \oplus (I + P)Y \oplus \dots)_X$. Here we denote by $(Y \oplus Y \oplus \dots)_X \to ((y_n)_{n=1}^n \circ y_n \in Y \text{ where } (\|y_n\|)_{n=1}^\infty \in Y^n$ tid ha operatoric companiori.

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Presented at the 2rd Symposiom on Functional Analysis and its Applications, Cratova, November 6-7, 1981; received James 5, 1982. 4 2 20 van (1 4 1989 Mathematics Subject Classification. Primary 46 A 10.